An Analysis of Core-guided Maximum Satisfiability Solvers Using Linear Programming

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Abstract

Many current complete MaxSAT algorithms fall into two categories: core-guided or implicit hitting set. The two kinds of algorithms seem to have complementary strengths in practice, so that each kind of solver is better able to handle different families of instances. This suggests that a hybrid might match and outperform either, but the techniques used seem incompatible. In this paper, we focus on PMRES and OLL, two core-guided algorithms based on max resolution and soft cardinality constraints, respectively. We show that these algorithms implicitly discover cores of the original formula, as has been previously shown for PM1. Moreover, we show that in some cases, including unweighted instances, they compute the optimum hitting set of these cores at each iteration. We also give compact integer linear programs for each which encode this hitting set problem. Importantly, their continuous relaxation has an optimum that matches the bound computed by the respective algorithms. This goes some way towards resolving the incompatibility of implicit hitting set and core-guided algorithms, since solvers based on the implicit hitting set algorithm typically solve the problem by encoding it as a linear program.

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1 Introduction

MaxSAT is the optimization version of SAT, in which we are given a set of *hard* clauses which must always be satisfied, as well as a set of weighted soft clauses, with the objective to find an assignment which minimizes the weight of the falsified soft clauses. Much like the case for SAT, the performance of MaxSAT solvers has been steadily improving over the past few years [5]. Two classes of algorithms have contributed significantly to this improvement: implicit hitting set (IHS) solvers [12, 14, 13, 6, 8] and core-guided solvers [18, 2, 24, 23, 22, 19]. Both are based on iteratively calling a SAT solver on formulas derived from the original MaxSAT instance and extracting unsatisfiable cores, but they are very different in their operation. IHS solvers exploit the hitting set duality of cores and correction sets (solutions)[26], and they try to build up a collection of cores that are enough to make the minimum hitting set match the optimum solution. Crucially, IHS solvers only ask the SAT solver to extract cores from subsets of the initial MaxSAT instance, which are all approximately equally hard. Core-guided solvers, on the other hand, reformulate the input instance with each core they discover so that it exhibits a higher lower bound. The reformulation generates ever more constrained formulas, which get harder and harder.

Despite their different approaches, both classes of algorithms are competitive, but they perform well in different families of instances. Hence, it would be desirable to understand exactly how they relate to each other and build algorithms with the strength of both. In that direction, Bacchus and Narodytska [7] showed that the cores discovered by the PM1 [18] algorithm correspond to a collection of cores of the original instance. Later, Narodytska and Bjørner [25] showed that for unweighted instances, PM1 actually discovers a hitting set of these cores of the original formula at every iteration. These results showed that there exists a close relationship between IHS and core-guided solvers.

Here, we focus on PMRES [24] and OLL [22]. Our contributions are as follows.

- We show that, like PM1, each core computed by PMRES and OLL corresponds to a set of cores of the original MaxSAT instance.
- We identify a condition for when the lower bound computed by PMRES or OLL matches the optimum hitting set of the set of cores of the original formula. This includes the case when the input instance is unweighted.
- We show that the hitting set problem over these cores can be formulated compactly as an integer linear program for both PMRES and OLL. Moreover, the linear relaxation of that ILP has a lower bound which is at least as great as the bound computed by PMRES or OLL, respectively.
- The linear program that we give is actually a subset of a higher level relaxation of that hitting set problem in the Sherali-Adams hierarchy [28].

The first two contributions match what has been done for PM1 previously, although our proofs are notably simpler, owing to the fact that the cores of PMRES and OLL have a much more regular structure than those of PM1. The latter two contributions provide further insight into the relationship between these core-guided algorithms and IHS. The LP formulation points the way to an algorithm that combines features of both core-guided and implicit hitting set solvers, since IHS solvers typically solve the hitting set problem with an ILP solver: any bounds computed by PMRES or OLL can be imported into IHS by way of this LP. The fact that this LP is a subset of a high level Sherali-Adams relaxation also shows IHS and core-guided solvers as being two extreme instantiations of the same algorithmic framework, where both solvers try to solve an implicit hitting set problem. But whereas IHS discovers only cores of the original formula and offloads solving of the hitting set problem to an external solver, PMRES very aggressively searches for a non-obvious set of new variables to add to the linear relaxation of the hitting set problem, in order to keep it as close as possible to the optimum integer solution, but places a great burden on the SAT solver. This suggests a more effective tradeoff could be found.

2 Background

In addition to the basics of MaxSAT, we also introduce necessary background on linear programming and weighted constraint satisfaction problems (WCSPs).

2.1 Satisfiability

A SAT formula ϕ in conjunctive normal form (CNF) is a conjunction of clauses and a clause is a disjunction of literals. We also view a CNF formula as a set of clauses and a clause as a set of literals. For a CNF formula F, we write $vars(F)$ for the set of all variables whose literals appear in the clauses of F. The Weighted Partial MaxSAT (WPMS) problem is a generalization of SAT to optimization. A WPMS formula is a triple $W = \langle H, S, w\rangle$ where H is a set of *hard* clauses, S is a set of *soft* clauses and $w : S \to \mathbb{R}_{\geq 0}$ is a cost function over the soft clauses. We also write $H(W) = H, S(W) = S, vars(W) = vars(H) \cup vars(S)$. For an assignment I over $vars(W)$, we overload notation to write $w(I) \triangleq \sum_{c \in S : I \vdash \neg c} w(c)$ for the cost of the soft clauses that I falsifies. The objective is to find an assignment I to $vars(W)$ such that all clauses in H are satisfied and the cost of the falsified soft clauses, i.e., $w(I)$, is minimized. We write $opt(W) \triangleq min_I w(I)$ for this value. A WPMS formula $\langle H, S, w\rangle$ with $w(c) = 1$ for all $c \in S$, is a partial MaxSAT formula. If, additionally, H is empty, it is a MaxSAT formula.

Two WPMS formula $W = \{H, S, w\}$ and $W' = \{H', S', w'\}$ are *equivalent* if for each assignment I to $vars(W)$ that satisfies H, we can extend it to an assignment I' to $vars(W')$ that satisfies H' and $w(I) = w'(I') + b$, for some constant b that is the same for all assignments. For example, $W = \{\emptyset, \{(x), (\overline{x})\}, w\}$, where $w((x)) = 5, w((\overline{x})) = 3$ is equivalent to $W' = \{\emptyset, \{(x)\}, w'\}$, where $w'((x)) = 2$, because the weight of all assignments differs by 3 in W, W'. This notion of equivalence is important in our subsequent analysis.

Given an unsatisfiable CNF formula F, a set $C \subset F$ is a core of F if C is unsatisfiable. If C is minimal by set inclusion, it is a MUS (minimal unsatisfiable subset) of F. Given a WPMS formula $W = \langle H, S, w\rangle$, a set $C \subseteq S$ is a core of W if $H \cup C$ is unsatisfiable. In the rest of this paper, we assume for simplicity that H is satisfiable and $H \cup S$ is unsatisfiable.

In the sequel, we make some assumptions without loss of generality. First, we assume that all soft clauses in a MaxSAT formula $W = \langle H, S, w\rangle$ are unit. If there exists a clause $c_i \in S$ which is not unit, we create the formula $W' = \langle H', S', w'\rangle$ with $H' = H \cup \text{cnf}(\neg c_i \iff b_i)$, $S' = S \cup \{(\overline{b}_i)\} \setminus \{c_i\}$, where b_i is a fresh variable, called the *blocking variable* for c_i, and $w'((\overline{b}_i)) = w(c_i)$, $w'(c) = w(c)$ for all $c \in S \cap S'$. We see that W is equivalent to W by noting that we can extend any assignment of W to W' by setting b_i so that it satisfies $b_i \iff \neg c_i$. Moreover, we assume that the unique literal in all soft clauses appears with negative polarity. If this does not hold, we can make it so by renaming. Because of this assumption, we identify each soft clause with the unique variable it contains and we use that literal to refer to it. Finally, we assume that there exist no soft clauses with cost 0, as we can remove those without affecting satisfiability or cost. However, we use the convention that $w(x) = 0$ for all positive literals and all negative literals of variables that do not appear in a soft clause. Given this convention, a WPMS instance can be written as $W = \langle H, w\rangle$, and S is implicitly $S = \{(\overline{x}_i \mid w(\overline{x}_i) > 0\}$. We use the two formulations interchangeably.

2.1.1 Solving WPMS

Most current SAT solvers have the ability to not only report SAT or UNSAT for a given formula, but also, given a partition of its clauses so that $\phi = \psi \cup \chi$, report a subset of χ such that $\psi \cup \chi$ is unsatisfiable. In terms of WPMS, it means a modern SAT solver can give a subset of S such that $H \cup S$ is unsatisfiable, i.e., a core of the WPMS formula. Because we have assumed that S contains negative unit clauses only, it follows that each core of W is a positive clause entailed by H.

The *implicit hitting set* (IHS) algorithm for WPMS [12, 14, 13, 6, 8] is based on the observation that the set of soft clauses $CS \subseteq S$ violated by a solution I is a hitting set of the set of all cores of W [26]. Hence, an optimal solution is a minimum hitting set of the cores of W. Hitting sets of all cores are called correction sets.

The IHS algorithm maintains an initially empty set of discovered cores $\mathcal{C}$ of W and a minimum hitting set of $\mathcal{C}$, $hs(\mathcal{C})$. If the SAT formula $H \cup (S \setminus hs(\mathcal{C}))$ is satisfiable, then its solutions are optimal solutions of W and $w(hs(\mathcal{C})) = w(W)$. Otherwise, a new core is extracted and added to $\mathcal{C}$ and the loop repeats. Actual implementations of the IHS algorithm in MaxHS [12] and LMHS [27, 9] contain many optimizations over this basic loop.

A *core-guided* algorithm for WPMS [18, 24, 23, 22, 19] is an iterative algorithm that generates a sequence of WPMS instances $W^0 = \langle H^0, w^0 \rangle = W, \ldots, W^m = \langle H^m, w^m \rangle$ and a sequence of lower bounds $lb^0 = 0 < lb^1 < \ldots < lb^m$ such that $H^i \models H^{i-1}$ for all $i \in [1, m]$ and W^0 is equivalent to W^i for all $i \in [1, m]$ and the weights of the assignments differ by lb_i, therefore $opt(W) = lb_i + opt(W^i)$. Moreover, in the last iteration it holds $opt(W^m) = 0$, so $opt(W) = lb^m$. In words, a core-guided algorithm generates a sequence of equivalent WPMS instances such that each successive instance is used to derive an increased lower bound for the original instance, while decreasing the cost of every solution by the same amount. The final instance admits a solution with zero weight, and each such solution of W^m is an optimal solution of W. All such solutions are solutions of the SAT formula $H^m \mid_0$, defined as $H^m \cup (\overline{x}) \mid w(x) > 0$, i.e., with all soft clauses made into hard clauses. In order to derive each successive instance W^{i+1} in the sequence, it extracts a core from W^i and uses it to transform it into W^{i+1} and increase the lower bound, hence the name core-guided. The algorithms we study here, PMRES and OLL, are core-guided algorithms. Following Narodytska and Bjørner [25], we call cores of W^i for $i > 0$ *meta cores*, or metas, to distinguish them from cores of the original formula W^0. We write m^i for the meta discovered at iteration i.

2.2 Linear programming and Weighted Constraint Satisfaction

An integer linear program (ILP) IP has the form $\min c^T x : Ax \geq b \wedge x \in \mathbb{Z}_{\geq 0}$, where x is a vector of n variables, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$. For a given x, if $Ax \geq b$, then it is a feasible solution of IP. We write $c(x) = c^T x$ for the *cost*[1] of x. We write $c(IP)$ for the cost of a feasible solution with minimum cost. The *linear relaxation* P of IP is the problem $\min c^T x : Ax \geq b \wedge x \in \mathbb{R}^n_{\geq 0}$, i.e., one where we relax the integrality constraint $x \in \mathbb{Z}^n_{\geq 0}$. This is called a linear program (LP). Linear programs have the *strong duality* property, namely that for every linear program P in the above form, there exists another linear program $P^D = \max b^T y : A^T y \leq c \wedge y \in \mathbb{R}^m_{\geq 0}$, with the property that $c_{P^D}(\hat{y}) \leq c_P(\hat{x})$ for every feasible solution $\hat{x}$ of P and $\hat{y}$ of P^D and $c_{P^D}(y^*) = c_P(x^*)$ for optimal solutions x^* and y^*. Given a feasible dual solution $\hat{y}$, the value $A_i^T y - c_i$, the slack of the dual constraint corresponding to the primal variable x_i, is called the *reduced cost* of x_i, denoted $rc_i(\hat{y})$. A necessary condition for optimality called *complementary slackness* links the two solutions: $x_i^* rc_i(y^*) = 0$, i.e., for each variable x_i, either it is zero or its corresponding dual constraint $(A_i y \leq c_i)$ is tight (has *zero slack*).

A Boolean Cost Function Network (CFN) is a pair $\langle V, D, C \rangle$, where V is a set of variables, D is a function mapping variables to domains, and C is a set of cost functions. If the domain of a variable v is binary, we write v for the value $v = 1$ and $\overline{v}$ for $v = 0$. Each cost function is a pair $\langle S, c \rangle$ where $S \subseteq V$ is its scope and c_S is a function $\prod_{x \in S} D(x) \to \mathbb{R}_{\geq 0} \cup \infty$. We assume there exists at most one cost function for each scope, so c_S is a shortcut for $\langle S, c_S \rangle$. An assignment I_S to a scope S is a function which maps every variable $x \in S$ to a value in $D(x)$. When we omit S, it means $S = V$. When convenient, we also use I to denote the set $\{v = a \mid I(v) = a, v \in V\} \cup \{v \neq b \mid I(v) \neq b, v \in V, b \in D(x)\}$. For a scope S and assignment I, $I_{\downarrow S}$ is the projection of I to S. $t(S)$ denotes all possible assignments to S.

We use the convention that for a cost function c_S, $c_S(I) = c_S(I_{\downarrow S})$, i.e., we implicitly project to S. For a CFN P, we write $c_P(I) = \sum_{c_S \in F} c_S(I)$. The Weighted Constraint Satisfaction Problem (WCSP) is to find an assignment I such that $c_P(I) < \infty$ and that

[1] We stick to the terminology of *weights* in MaxSAT and *costs* in ILP and WCSP, even though they serve the same purpose.

minimizes c_P. The term WCSP is often used to refer both to the underlying CFN and to the optimization problem, and we do the same here. Additionally, we assume the existence of a unary cost function $c_{\{v\}}$ (abbreviated as c_v) for every variable $v \in V$ and a nullary cost function $c_\varnothing$, which is a lower bound for c_P, becayse all costs are non-negative. A CSP is a WCSP in which the domain of all cost functions is $\{0, \infty\}$.

A WCSP $P = \langle V, C\rangle$ can be formulated as the following ILP:

$$\min \sum_{c_S \in C, l \in t(S)} c_S(l) x_{Sl} \tag{1}$$

$$\text{s.t.} \tag{2}$$

$$x_{\{v\},a} = \sum_{l \in t(S): v=a \in l} x_{Sl} \qquad \forall v \in V, a \in D(v), c_S \in C \tag{3}$$

$$\sum_{a \in D(v)} x_{\{v\},a} = 1 \qquad \forall v \in V \tag{4}$$

$$x_{Sl} \in \mathbb{Z}_{\geq 0} \qquad \forall v \in V, c_S \in C, l \in t(S) \tag{5}$$

The linear relaxation of (1)– (5) defines the *local polytope* of P. A dual feasible solution of the local polytope LP has a particular interpretation: it defines a *reformulation* of the WCSP. A reformulation can be seen as a set of operations on a WCSP P that create a new WCSP $\hat{P}$ with modified costs, but $c_P(I) = c_{\hat{P}}(I)$ for all I. Therefore, a reformulation is said to *preserve equivalence*. This notion of equivalence is identical to the equivalence preserved by core-guided algorithms, with the primary difference being that the lower bound is explicitly represented in a WCSP in $c_\varnothing$. These operations can intuitively be thought of as moving cost among cost functions:

- Extention: $ext(v = a, c_S, \alpha)$, with $v \in S, a \in D(v)$. This subtracts cost α from $c(\{v\}, a)$ and adds it to $c(S, l)$ for all tuples $l \in t(S) : (v = a) \in l$. To see the correctness of this, consider the subset of the objective function $c_v(a)x_{\{v\},a} + \sum_{l \in t(S):(v=a) \in l} c_S(l)x_{Sl}$, as well as constraint (3). Since $x_{\{v\},a}$ is equal to the sum, the value of the objective remains unchanged by adding α to one and subtracting it from the other.
- Projection: $prj(c_S, v = a, \alpha)$, with $v \in S, a \in D(v)$. This is the same as $ext(v, c_S, -\alpha)$.
- Nullary projection: $prj_0(c_S, w)$. This subtracts cost α from each tuple $l \in t(S)$ and moves it to $c_\varnothing$. This is justified because $\sum_{l \in t(S)} x_{Sl} = 1$ and the cost of $c_\varnothing$ is a constant in the objective function.

Because these operations preserve equivalence, they are called Equivalence Preserving Transformations (EPTs). A valid set of EPTs ensures that all cost functions are non-negative everywhere, but there are valid sets of EPTs for which any sequence of performing them leaves intermediate negative costs. A valid set of EPTs can be mapped to a feasible dual solution of the local polytope LP and vice versa. A set of EPTs which achieves the greatest increase in $c_\varnothing$, and hence the lower bound, can be mapped to an optimal dual solution of the local polytope LP [11]. Given a dual solution, the cost of each tuple $l \in t(S)$ is given by the reduced cost of the variable x_{Sl}.

For a WCSP P, let $Bool(P)$ be the CSP (not weighted) defined by accepting exactly those tuples which have cost 0, i.e., changing all costs which are greater than 0 to ∞. Let $\hat{P}$ be a reformulation of P. A consequence of complementary slackness is that if $\hat{P}$ is an optimal reformulation, then $Bool(\hat{P})$ has a non-empty arc consistency closure [11, 15], in which case it is said that $\hat{P}$ is virtually arc consistent (VAC). This is not a sufficient condition for optimality, however. Conversely, if $\hat{P}$ is not VAC, therefore $Bool(\hat{P})$ has an empty arc consistency closure, there exists a reformulation with a higher $c_\varnothing$.

3 PMRES

The PMRES algorithm is a core guided solver which was introduced by Narodytska and Bacchus [24] and is implemented primarily in the EVA solver. We describe it briefly here. In this description, we use the view of WPMS as hard and soft clauses, rather than hard clauses and an objective, because the transformations performed by PMRES temporarily violate the assumptions that allow us to take this alternative view. However, these assumptions are always restored at the end of each iteration.

3.1 Max-Resolution

Max resolution [20] is a complete inference rule for MAXSAT [10]. It consists of the following rule on soft clauses, in which the conclusions replace the premises:

$$\frac{\begin{array}{c}(A \vee x, w)\\(B \vee \overline{x}, w)\end{array}}{\begin{array}{c}(A \vee B, w)\\(A \vee x \vee \overline{B}, w)\\(B \vee \overline{x} \vee \overline{A}, w)\end{array}}$$

The first clause in the conclusions is equivalent to what resolution derives. The latter two are called compensation clauses, as they compensate for the cost of assignments which do not falsify the conclusion $A \vee B$ but falsify one of the discarded premises. Depending on the exact form of A and B, the compensation "clauses" may not actually be in clausal form and would have to be converted to a set of clauses each. We ignore this complication here, as our presentation of PMRES mostly avoids this case.

The rule is generalized to clauses with different costs $w_1 > w_2$ by cloning the heavier clause into clauses with costs w_2 and $w_1 - w_2$. When one of the clauses is hard, e.g., $w_1 = \infty$, we keep it in the conclusions.

Max resolution has the property that if W and $\hat{W}$ are the formulas before and after application of the rule, then they are equivalent.

3.2 Max-Resolution with cores

PMRES uses the specialization of this rule for a binary clause and a unit clause, i.e., $|A| = 1, B = \emptyset$.

$$\frac{\begin{array}{c}(A \vee x, w)\\(\overline{x}, w)\end{array}}{\begin{array}{c}(A, w)\\(\overline{x} \vee \overline{A}, w)\end{array}}$$

As a core-guided solver, PMRES is an iterative algorithm and the first step in each iteration is to extract a meta core from W^i, or terminate if $H^i \cup S^i$ is satisfiable. Suppose that the meta is $m^i = \{b_1^i, b_2^i, \ldots, b_{r^i}^i\} \subseteq S^{i-1}$ and $w_{\min}^i = \min_{b_j \in C} c^i(b_j^i)$. This implies the presence of the soft clauses $(\overline{b}_1^i, w_1), \ldots, (\overline{b}_{r^i}^i, w_{r^i})$. PMRES first splits each soft clause $(\overline{b}_j^i, w')$ with $w' > w_{\min}^i$ into $(\overline{b}_j^i, w_{\min}^i)$ and $(\overline{b}_j^i, w' - w_{\min}^i)$. This temporarily violates our assumption that each soft clause contains a unique literal, but as we will see, this invariant

$(b_1 \vee b_2 \vee b_3 \vee b_4)$
$(b_5 \vee b_2)$
$(b_5 \vee b_3 \vee b_4)$

Figure 1 Cores of the instance used in the running example

is restored before the next iteration starts. In the next step, it adds to H^{i+1} the hard clause corresponding to C using the CNF encoding of $(b^i_1 \vee d^i_1), (d^i_1 \iff b^i_2 \vee d^i_2), \ldots (d^i_{r^i-2} \iff b^i_{r^i-1} \vee d^i_{r^i_1}), (d^i_{r^i-1} \iff b^i_r)$, where $d^i_1, \ldots, d^i_{r-1}$ are fresh variables. It is clear that we can recover the clause $(b^i_1 \vee \ldots \vee b^i_r)$ by resolving (not with max-resolution, as the clauses are all hard) the first two clauses on d^i_1, then on d^i_2, and so on, therefore the encoding and the clause are equivalent. PMRES then applies max-resolution as follows:

Premises		Conclusions
$(b^i_1 \vee d_1, w^i_{\min})$	$(\overline{b}^i_1, w^i_{\min})$	$(d_1, w^i_{\min}), \boxed{(\overline{b}^i_1 \vee \overline{d}_1, w^i_{\min})}$
$(d_1, w^i_{\min})$	$(\overline{d}_1 \vee b^i_2 \vee d_2, w^i_{\min})$	$(b^i_2 \vee d_2, w^i_{\min}), (\overline{(b^i_2 \vee d_2)} \vee d_1, w^i_{\min})$
$\vdots$		
$(b^i_{r-1} \vee d_{r-1}, w^i_{\min})$	$(\overline{b}^i_{r-1}, w^i_{\min})$	$(d_{r-1}, w^i_{\min}), \boxed{(\overline{b}^i_{r-1} \vee \overline{d}_{r-1}, w^i_{\min})}$
$(d_{r-1}, w^i_{\min})$	$(\overline{d}_{r-1} \vee b^i_r, w^i_{\min})$	$(b^i_r, w^i_{\min}), (\overline{b}^i_r \vee d_{r-1}, w^i_{\min})$
$(b^i_r, w^i_{\min})$	$(\overline{b}^i_r, w^i_{\min})$	$\boxed{(\Box, w^i_{\min})}$

The non-clausal constraints in light gray are tautologies and can be discarded. For example, by $(d^i_1 \iff b^i_2 \vee d^i_2)$, $(\overline{b^i_2 \vee d^i_2}) \vee d^i_1$ is equivalent to $(\overline{d}^i_1 \vee d^i_1)$, a tautology. The clauses in gray are used as input for the next max-resolution step. The framed clauses are new soft clauses that are kept for the next iteration. Since they are not unary, they are reified using fresh variables and converted to unit soft clauses, e.g., $f \iff b^i_1 \wedge d^i_1$ and $(\overline{f}, w)$, where f is the fresh variable. Finally, the empty soft clause $(\Box, w_{\min})$ is used to increase the lower bound for the next iteration by $w_{\min}$.

Consider now a clause $(\overline{b}^i_j, w')$ that was split into two clones $(\overline{b}^i_j, w_{\min})$ and $(\overline{b}^i_j, w' - w_{\min})$. The former is consumed by max-resolution, therefore the invariant that each soft clause contains a unique literal is restored. This also allows us to implement the cloning process as a simple update: $w^{i+1}(b_j) = w^i(b_j) - w_{\min} = w' - w_{\min}$. If it happens that $w' = w_{\min}$, we maintain by the previously mentioned convention that $w^{i+1}(b_j) = 0$.

In the following, we write H^i_R for the formula consisting only of the clauses introduced by PMRES, therefore $H^i = H \cup H^i_R$. We also write F^i and D^i for the set of all variables, introduced to reify soft clauses (e.g. f above) or to encode the meta core clause (the d^i_j variables above), respectively. It has also been previously noted [25, 3] that the conjunction of the definitions of the F and D and the clauses $(b^i_1 \vee d^i_1)$ define a monotone circuit, with a binary gate corresponding to each $v \in F^i \cup D^i$, an unnamed $\vee$ gate corresponding to the clause $(b^i_1 \vee d^i_1)$, and an implicit $\wedge$ gate whose inputs are the unnamed $\vee$ gates, which is the output of the circuit.

▶ **Example 1** (Running Example). Consider an instance W with 5 soft clauses with cost 1 each and corresponding literals $b_1, \ldots, b_5$, and the cores shown in Figure 1. We show a run of PMRES in Figure 2 (for readability, we show the objective function rather than the set of soft clauses) that discovers first the core $(b_1 \vee b_2 \vee b_3 \vee b_4)$. It increases the lower bound by 1, adds the variables $D^1 = \{d^1_1, d^1_2, d^1_3\}$ and $F^1 = \{f^1_1, f^1_2, f^1_3\}$, defined as shown in the row corresponding to iteration 1. Since weights are unit, all original variables except b_5 disappear

Iteration	Meta	New clauses	Objective
1	$\{b_1, b_2, b_3, b_4\}$	$d_1^1 \iff b_2 \vee d_2^1$, $d_2^1 \iff b_3 \vee d_3^1$,	
		$d_3^1 \iff b_4$,	
		$f_1^1 \iff b_1 \wedge d_1^1$, $f_2^1 \iff b_2 \wedge d_2^1$,	
		$f_3^1 \iff b_3 \wedge d_3^1$	$1 + b_5 + f_1^1 + f_2^1 + f_3^1$
2	$\{f_2^1, b_5\}$	$d_1^2 \iff b_5$	
		$f_1^2 \iff b_7 \wedge d_1^2$	$2 + f_1^1 + f_3^1 + f_1^2$

Figure 2 PMRES on the running example

from the objective. In the next iteration, PMRES discovers the meta $\{b_5, f_2^1\}$, increases the lower bound to 2, and introduces the variables d_1^2 and f_1^2. In the next iteration, the instance is satisfiable. One of the possible solutions is b_4, b_5, with cost 2, which matches the lower bound.

3.3 Cores and Hitting Sets of PMRES

We first observe that the f^i and d^i variables created on iteration i are functionally dependent on the b^i variables. Therefore, the formula H^i generated after the ith iteration is logically equivalent to H, i.e., every solution of H can be extended to exactly one solution of H^i.

▶ **Lemma 2.** *There exists a set $\mathcal{C}^i$ such that m^i is a core of H^i if and only if for each $c \in \mathcal{C}^i$, c is a core of ϕ.*

Proof. The set $\mathcal{C}^i$ can be derived from m^i and H_R^i by forgetting the variables f and d that were introduced by PMRES. More concretely, let $E^0 = \{m^i\}$. If there exists $c \in E^j$ such that $f \in c$ and f was introduced by PMRES and defined as $f \iff b \wedge d$, we set $E^{j+1} = E^j \setminus \{c\} \cup \{c \setminus \{f\} \cup \{b\}, c \setminus \{f\} \cup \{d\}\}$, i.e., we replace c by two clauses which have b and d, respectively, instead of f. If there exists $c \in E^j$ such that $d \in c$ and d was introduced by PMRES and defined as $d \iff b \vee d'$, we set $E^{j+1} = E^j \setminus \{c\} \cup \{c \setminus \{d\} \cup \{b, d'\}\}$, i.e., we replace d by b, d' in c. The process eventually terminates because it removes one reference to a variable introduced by PMRES and replaces it by a variable corresponding to a gate at a deeper level of the Boolean circuit defined by H_R^i, hence all variables must eventually be original variables of W^0. It is also confluent because the choice of variable to forget does not hinder other choices.

Since both forgetting variables and introducing functionally defined variables are satisfiability-preserving operations, we have $m^i \wedge H_R^i \models \mathcal{C}^i$ and $\mathcal{C}^i \models m^i \wedge H_R^i$. ◀

▶ **Lemma 3.** *Let $hs \subseteq S$. Then hs as an assignment can be extended to a solution of H_R^i if and only it is a hitting set of $\mathcal{C}_\cup^i$.*

Proof. This follows from lemma 2.

($\Rightarrow$) hs satisfies H_R^i, hence it satisfies all clauses in $\mathcal{C}^i$, which are cores, so it hits all the cores.

($\Leftarrow$) hs is a hitting set of $\mathcal{C}^i$, hence it satisfies all the corresponding clauses, hence it satisfies H_R^i. ◀

In the following, let $\mathcal{C}_\cup^i = \cup_{j \in [1,i]} \mathcal{C}^j$.

▶ **Observation 4.** *$\langle H_R^i, w^0 \rangle$ and $\langle H_R^i, w^i \rangle$ are equivalent.*

Proof. Consider $H_0 = \langle H_R^i, w^0 \rangle$. We know that m^0 is a core of H_0. By applying max resolution to m^0 as described in section 3.2, we get new variables and soft clauses. But these new variables are defined identically to the variables PMRES introduced to get H_R^1, which is a subset of H_R^i. Hence, we can identify them. By correctness of PMRES, we get that $\langle H_R^i, w^1 \rangle$ is equivalent to $\langle H_R^i, w^0 \rangle$. We apply the same argument inductively to complete the proof. ◀

▶ **Corollary 5.** *The WPMS $W_i^{hs} = \langle H_R^i, w^i \rangle$ encodes the minimum hitting set problem over $\mathcal{C}_\cup^i$, with weights shifted by lb^i. Hitting sets with cost lb^i, if they exist, are solutions of W_{hs}^i that use only soft clauses with soft 0.*

Proof. From Lemma 3, $\langle H_R^i, w^0 \rangle$ encodes minimum hitting set over $\mathcal{C}_\cup^i$. From Observation 4, $\langle H_R^i, w^0 \rangle$ and $\langle H_R^i, w^i \rangle$ are equivalent, therefore W_i^{hs} encodes minimum hitting set over $\mathcal{C}_\cup^i$.

The second part follows from the fact that, for any assignment I, $w^0(I) = lb_i + w^i(I)$, so if $w^0(I) = lb_i$, then $w^i(I) = 0$. ◀

Let us denote by $H_R^i\mid_0$ the formula H_R^i with all variables x such that $w(x) > 0$ set to false so that all models of $H_R^i\mid_0$ are minimum hitting sets of $\mathcal{C}^i$. Therefore if $H_R^i\mid_0$ is satisfiable, the bound computed by PMRES matches the cost of the minimum hitting set of $\mathcal{C}_\cup^i$.

▶ **Lemma 6.** *If W is a PMS instance, $H_R^i\mid_0$ is satisfiable for all iterations i of* PMRES.

Proof.
- All variables in D have cost 0.
- Moreover, all variables which appear in any meta have cost 0, because it is moved away by max-resolution.
- Therefore, all variables in $b_1^j, \ldots, b_{r^j}^j$ for $j \in [1, i]$ have zero cost.

We construct a solution to $H_R^i\mid_0$ by setting to false all variables which are inputs to false $\wedge$-gates (which is done by unit propagation), then we set variables to true by traversing metas in reverse chronological order:

1. For m^i, we pick the first variable in $b_1^j, \ldots, b_{r^j}^j$ and set it to true. We set all variables in F^i and D^i to false (the former is required for m^i because, as the last discovered core, all variables in F^i have non-zero weight.
2. Supposing we have satisfied all metas $m^{j+1}, \ldots, m^i$, consider m^j. Suppose that $0 \leq q < |m^j|$ variables in F^j that have been set to true by previous steps, with indices $P^j = \{p_1, \ldots, p_q^j\}$. For simplicity of notation, assume that if P^j is empty, then $p_q^j = 0$. Then we set to true the variables $b_r^j \mid r \in P^j$ as well as $b_{p_q+1}^j$, and set the rest to false. When $p_q^j = 0$, this reduces to setting the first variable in b_1^j to true.
   a. This assignment satisfies the constraints introduced in H_R^j.
   b. Moreover, all the variables that appear in m^j have cost 0 after the j^{th} iteration. Therefore they cannot appear in any meta discovered in iterations $j+1, \ldots, i$ and the assignment we have chosen here does not contradict the assignments chosen in iterations $j+1, \ldots, i$.

◀

We can see where the proof of Lemma 6 breaks when applied to WPMS: the assertion 2b does not hold, because a variable whose cost has not been reduced to 0 may appear in later metas and our procedure may therefore create a conflicting assignment.

Iteration	Core	New clauses	Objective
1	$\{b_1, b_2, b_3, b_4\}$	$d_1^1 \iff b_2 \vee d_2^1$, $d_2^1 \iff b_3 \vee d_3^1$, $d_3^1 \iff b_4$, $f_1^1 \iff b_1 \wedge d_1^1$, $f_2^1 \iff b_2 \wedge d_2^1$, $f_3^1 \iff b_3 \wedge d_3^1$	$1 + b_2 + 2b_3 + 3b_4 + 5b_5 +$ $f_1^1 + f_2^1 + f_3^1$
2	$\{f_2^1, b_5\}$	$d_1^2 \iff b_5$ $f_1^2 \iff f_2^1 \wedge d_1^2$	$2 + b_2 + 2b_3 + 3b_4 + 4b_5 +$ $f_1^1 + f_3^1 + f_1^2$
3	$\{b_3, b_4, b_5\}$	$d_1^3 \iff b_4 \vee d_2^3$, $d_2^3 \iff b_5$ $f_1^3 \iff b_3 \wedge d_1^3$, $f_2^3 \iff b_4 \wedge d_2^3$	$4 + b_2 + b_4 + 2b_5 +$ $f_1^1 + f_3^1 + f_1^2 + 2f_1^3 + 2f_2^3$
4	$\{b_2, b_5\}$	$d_1^4 \iff b_5$ $f_1^4 \iff b_2 \wedge d_1^4$	$5 + b_4 + b_5 +$ $f_1^1 + f_3^1 + f_1^2 + 2f_1^3 + 2f_2^3 + f_1^4$

Figure 3 PMRES on the running example with modified, non-unit weights.

▶ **Example 7** (PMRES on a weighted formula)**.** Consider the running example, but with the modified weights $(1, 2, 3, 4, 5)$, respectively. We assume the same trail as shown in figure 2, and show in figure 3 the modified execution. After the first two iterations the lower bound will be 2, as shown. The optimum hitting set is $\{b_2, b_3\}$ with cost 5, so the lower bound does not match the optimum. Indeed, $H_R^2\mid_0$ is unsatisfiable: the clause $(f_2^1 \vee b_5)$ can only be satisfied by f_2^1, because $w^2(b_5) > 0$. But $f_2^1 \iff b_2 \wedge (b_3 \vee b_4)$ and $w^2(b_2) > 0, w^2(b_3) > 0, w^2(b_4) > 0$, therefore f_2^1 is forced to false. Hence, PMRES has to perform more iterations before matching the bound of the hitting set. A possible trail finds the metas $\{b_3, b_4, b_5\}$ and $\{b_2, b_5\}$ (which also happen to be cores of W^0), as shown.

We are now ready to state the main result of this section.

▶ **Theorem 8.** *For a PMS instance, at each iteration,* PMRES *computes an optimum hitting set of* $\mathcal{C}_\cup^i$.

Proof. Follows from Lemma 2, Corollary 5, and Lemma 6. ◀

For a WPMS instance, we can get a weaker result: since cores of $H_R^i\mid_0$ are also cores of $H^i\mid_0$, we can extract cores of $H_R^i\mid_0$, which are metas of W until it becomes satisfiable, at which point the bound is a hitting set of $\mathcal{C}_\cup^i$. It is not clear if that is a desirable thing to do from a performance perspective.

3.4 PMRES and Linear Programming

In this section, we prove the following.

▶ **Theorem 9.** *There exists an integer linear program ILP_P^i which (1) is logically equivalent to the minimum hitting set problem with sets $\mathcal{C}_\cup^i$, (2) has size polynomial in $|H_R^i|$, and (3) whose linear relaxation has an optimum which matches that derived by* PMRES.

Given the results of section 3.3, (1) is easy to show, since we can generate the set $\mathcal{C}_\cup^i$, then write the hitting constraint for each set in $\mathcal{C}_\cup^i$, and use w^0 as the objective. Call this ILP_{hs}^i. But ILP_{hs}^i may be exponentially larger than H_R^i. It is not much harder to show that we can achieve (1) and (2). As Corollary 5 shows, H_R^i is logically equivalent to that hitting set problem, so we can replace the constraints of ILP_{hs}^i by H_R^i (i.e., by the standard encoding of clauses to linear constraints) and get an equivalent problem. Call that ILP_R^i and its linear relaxation LP_R^i.

However, we can see that LP_R^i is weak, specifically, that $c(LP_R^i) < c(ILP_R^i)$.

▶ **Example 10** (Running example, continued). Consider the ILPs ILP_{hs}^2 and ILP_R^2 corresponding to the hitting set problems for the 2^{nd} iteration of PMRES on the instance W in our running example. The optimum of both ILP_{hs}^2 and ILP_R^2 is 2, as expected, but the optimum of LP_R^2 is only 1.5.

In this specific example, since we have integer costs, the bound of the linear relaxation allows us to derive a bound of 2 for ILP_R^2, but in general we can get an arbitrarily large difference. This is not surprising in general, but the fact that PMRES does compute an optimal hitting set at each iteration suggests that we should be able to do better. This is the objective of this section.

To construct an LP that meets the requirement of the theorem, we give a WCSP and its reformulation, which yield an LP (the local polytope) and a dual solution (one which is created from the formulation), as described in section 2.2. The result could be proved by directly giving an appropriate LP and dual solution, and proving the result on that, but it would be more cumbersome and would lack the existing intuitive understanding that has been developed in WCSP of dual solutions as reformulations.

Proof of theorem 9. We will give first a WCSP P^i which admits the same solutions as H_R^i and has unary costs such that its feasible solutions have the same cost as the hitting set problem entailed at iteration i of PMRES. This means that the optimum solution of P^i matches the minimum hitting set of $\mathcal{C}_\cup^i$. Further, we show that its linear relaxation $LP(P^i)$ admits a dual feasible solution whose cost matches the bound computed by PMRES. We give this dual solution as a sequence of equivalence preserving transformations of P^i, using the results presented in section 2.2. That linear program, $LP(P^i)$, satisfies the requirements of the theorem.

We first define P^i. The high level idea is that the we encode the objective function of ILP_R^i directly as unary costs, and each meta using the well known decomposition into ternary constraints. The d variables have exactly the same semantics as the auxiliary variables used in that decomposition. The corresponding f variable corresponds to a single tuple of these ternary constraints, so we add an f variable to each ternary constraint in order to capture the cost of that ternary tuple into a unary cost. More precisely, let $P^0 = \emptyset$. At iteration i, where the core discovered is $\{b_1^i, b_2^i, \ldots, b_r^i\} \subseteq S^{i-1}$, P^i is defined as P^{i-1} and additionally the following variables and cost functions:

- 0/1 variables $b_1, \ldots, b_n$, d_j^i, f_k^i, corresponding to the propositional variables of the same name in W^i.
- Unary cost functions with scope b_i for each $b_i \in vars(W^0)$, with $c_{b_i}(0) = 0, c_{b_i}(1) = c^0(b_i)$
- A ternary cost function with scope $\{b_1^i, d_1^i, f_1^i\}$ where each tuple that satisfies $b_1^i \vee d_1^i$ and $f_1^i \iff d_1^i \wedge b_1^i$ has cost 0 and the rest have infinite cost.
- Quaternary cost functions with scope $\{b_j^i, d_{j-1}^i, d_j^i, f_j^i\}$, for $j \in [2, r-2]$, where each tuple that satisfies $d_{j-1}^i \iff d_j^i \vee b_j^i$ and $f_j^i \iff d_j^i \wedge b_j^i$ has cost 0 and the rest have infinite cost.
- A binary cost function with cost 0 for each tuple that satisfies $d_{r-1}^i = b_r^i$ and infinite cost otherwise.

It is straightforward to see that P^i is equivalent to ILP_R^i: (i) they have the same set of variables, (ii) the only costs in P^i are in unary cost functions, so the objective functions are the same, (iii) the quaternary cost functions satisfy, by construction, the clauses included in the scope of these functions, and (iv) each clause is present in one cost function. Therefore, solutions of P^i are hitting sets of $\mathcal{C}^i_\cup$ and the cost of each solution matches the cost of the corresponding hitting set.

It remains only to show that the LP optimum of $relax(P^i)$ matches that produced by PMRES. We show a slightly stronger result, namely that there exists a sequence of EPTs such that in P^i, not only does the bound match that produced by PMRES, but the unary costs of each variable match the weights computed by PMRES. We show this by induction on the number of iterations. At iteration 0, this holds trivially, as the bound is 0 for both P^0 and PMRES and the unary costs match the weights by construction. Suppose it holds at iteration $k-1$. Then, the core at iteration k is $\{b^k_1, b^k_2, \ldots, b^k_r\} \subseteq S^{k-1}$. The EPT $ext(b^k_1, \{b^k_1, d^k_1, f^k_1\}, w^k_{\min})$ enables the EPTs $prj(\{b^k_1, \overline{d}^k_1, f^k_1\}, f^k_1, w^k_{\min})$ and $prj(\{b^k_1, d^k_1, f^k_1\}, \overline{d}^k_1, w^k_{\min})$. For $j \in [2, r^k-2]$, in addition to extending cost from b^k_j, we also extend from $\overline{d}^k_{j-1}$, which has just received this amount of cost: $ext(b^k_j, \{b^k_j, d^k_{j-1}, d^k_j, f^k_j\}, w^k_{\min})$ and $ext(\overline{d}^k_{j-1}, \{b^k_j, \overline{d}^k_{j-1}, d^k_j, f^k_j\}, w^k_{\min})$, which enable $prj(\{b^k_j, d^k_{j-1}, d^k_j, f^k_j\}, f^k_j, w^k_{\min})$ and $prj(\{b^k_j, d^k_{j-1}, d^k_j, f^k_j\}, \overline{d}^k_j, w^k_{\min})$. Finally, after $j = r-2$, we are left with $w^k_{\min}$ in $\overline{d}^k_{r-1}$. Using $d^k_{r-1} \iff b^k_r$, we move cost from b^k_r to d^k_{r-1} (specifically: $ext((, b^k_r, \{b^k_r, d^k_r\})w^k_{\min}$, then $prj(\{b^k_r, d^k_r\}, d^k_r, w^k_{\min})$. Since both d^k_r and $\overline{d}^k_r$ have cost $w^k_{\min}$, we can apply $prj_0(d^k_r, w^k_{\min})$ to increase the lower bound by $w^k_{\min}$.

After these EPTs, not only is the lower bound increased by $w^k_{\min}$, but the variables $b^k_1, \ldots, b^k_r$ have their cost decreased by $w^k_{\min}$, the variables $f^k_1, \ldots, f^k_{r-1}$ receive cost $w^k_{\min}$, and the variables $d^k_1, \ldots, d^k_{r-1}$ stay at 0. This matches the effects of PMRES, as required by the inductive hypothesis. ◀

▶ **Example 11.** We move away from our running example here, as showing and explaining all the cost moves would be tedious and space consuming. Instead, we give a small example with the core $\{b^1_1, b^1_2, b^1_3\}$ in figure 4. All variables of this core have uniform weight w. We show how the EPTs remove cost from b^1_1, b^1_2, b^1_3 and move it to f^1_1, f^1_2 and $c_\varnothing$, leaving all other cost functions unchanged, even though they were used to make the cost moves possible. The increase in $c_\varnothing$ comes from a nullary projection from b^1_3.

Note that theorem 9 does not prove that the optimum of (P^i) is identical to that of PMRES at iteration i, but only that it is at least as high, as the following example shows.

▶ **Example 12** (Running example, continued)**.** After iteration 2, in the running example, unit propagation alone detects the core $\{b_3, b_4, b_5\}$. This means that when we set these variables to false because their weight is non-zero, unit propagation generates the empty clause.

Let $\hat{P}^i$ be the reformulation of P^i given by theorem 9. Then H^i_R and $\hat{P}^i$ have the same costs/weights. $H^i_R\mid_0$ is constructed from H^i_R in the same way as $Bool(P^i)$ is constructed from $Bool(P)$: by making each non zero cost (weight) into an infinite cost (weight). so $H^i_R\mid_0$ admits the same solutions as $Bool(\hat{P}^i)$. Moreover, each clause of $H^i_R\mid_0$ is contained in at least one constraint of $\hat{P}^i$, therfore arc consistency on $Bool(\hat{P}^i)$ is at least as strong as unit propagation on $H^i_R\mid_0$. And since the core $\{b_3, b_4, b_5\}$ is not satisfied, the arc consistency closure of $Bool(\hat{P}^i)$ is empty, therefore its bound can be improved further.

On the other hand, there is no reason to expect that the the optimum of (P^i) will necessarily be higher than the bound computed by PMRES. For example, if $H^i_R\mid_0$ has no cores that can be detected by unit propagation, the argument of example 12 does not apply.

4 OLL

OLL [22] is probably the most relevant core-guided algorithm currently, since solvers based on it, like RC2 [19] and CASHWMaxSAT-CorePlus [21] have done very well in recent MaxSAT

b_1^1	d_1^1	f_1^1	0	1	2
0	1	0	0		
1	0	0	0	w	0
1	1	1	0	w	0

b_2^1	d_1^1	d_2^1	f_2^1	0	3	4
0	0	0	0	0	w	0
0	1	1	0	0		
1	1	0	0	0	w	0
1	1	1	1	0	w	0

b_1^1	0	1
0	0	
1	w	0

d_1^1	0	2	3
0	0	w	0
1	0		

f_1^1	0	2
0	0	
1	0	w

b_2^1	0	3
0	0	
1	w	0

b_3^1	0	4	5
0	0	w	0
1	w	w	0

f_2^1	0	4
0	0	
1	0	w

$c_\varnothing$	0	5
	0	w

Figure 4 The evolution of cost functions that leads to the increase of the lower bound by w for the core $\{b_1^1, b_2^1, b_3^1\}$. Each table shows a cost function and how it evolves after each EPT. We omit the rows which would violate one of the clauses introduced by PMRES, as infinity absorbs all costs, so they are unaffected by EPTs. The column 0 gives the initial costs. Subsequent columns give the state of each cost function after all EPTs to that point. Only points in the sequence which affect a given cost function are given in the corresponding table. Since $d_2^1 = b_3^1$ for this core, we simplify the problem here and replace occurrences of d_2^1 by b_3^1 rather than include an extra binary cost function to enforce their equality. The sequence is 1: $ext(b_1^1, \{b_1^1, d_1^1, f_1^1\}, w)$, 2: $prj(\{b_1^1, d_1^1, f_1^1\}, \overline{d}_1^1, w)$ and $prj(\{b_1^1, d_1^1, f_1^1\}, f_1^1, w)$, 3: $ext(b_2^1, \{b^1_{,} d_1^1, b_3^1, f_1^1\}, w)$ and $ext(\overline{d}_1^1, \{b^1_{,} d_1^1, b_3^1, f_1^1\}, w)$, 4: $prj(\{b^1_{,} d_1^1, b_3^1, f_1^1\}, \overline{b}_3^1, w)$ and $prj(\{b^1_{,} d_1^1, b_3^1, f_1^1\}, f_2^1, w)$, 5: $prj_0(b_3^1, w)$

evaluations [5].

4.1 MaxSAT with soft cardinality constraints

OLL is an iterative algorithm, similar to PMRES. For the purposes of this discussion, it only differs in how it processes each meta that it finds. At iteration i, given the meta $m^i = \{b_1^i, b_2^i, \dots, b_{r^i}^i\} \subseteq S^{i-1}$, it adds fresh variables $o_1^i, \dots, o_{r^i-1}^i$ and constraints $o_j \iff \sum_{k=1}^{r^i} b_k^i > j$, then decreases the weight of each variable in m^i by $w_{\min}^i$, increases the lower bound by $w_{\min}^i$, and sets the weight of the fresh variables $o_1^i, \dots, o_{r^i-1}^i$ to $w_{\min}^i$. The o variables are called *sum variables.*

4.1.1 OLL with implied cores

We use here a minor modification of OLL, which we denote OLL′. In this variant, before processing a meta at iteration i, each sum variable o_k^j, $j < i, k \in [2, r^j - 1]$ is replaced by $o_{k'}^j$ where $k' < k$ is the lowest index for which $w(o_{k'}^j) > 0$. This is sound because $o_k^j \rightarrow o_{k'}^j$ for all $k' < k$, which can be written as $\neg o_k^j \vee o^{k'}$. We can resolve the meta at iteration i with this clause to effectively replace o_k^j by $o_{k'}^j$. This procedure can be repeated as long as it results in a meta with non-zero minimum weight, although that step is not required for the results we obtain next.

We argue that OLL′ matches the behaviour of a realistic implementation like RC2, when used with an assumption-based solver such as MINISAT [16] or a derivative like GLUCOSE [4]. In order to extract a core with MINISAT, RC2 asserts the negation of all literals which may appear in a core as assumptions. These literals are passed to MINISAT as a sequence. MINISAT returns a subset of these literals as a core. Crucially, MINISAT immediately propagates each assumption in sequence and never returns in a core a literal which is implied by earlier assumptions. Therefore, if the literals of the soft clauses introduced by OLL are given in the order $\langle o^i_1, \dots o^i_{r^i} \rangle$, we get from $o^i_{j+1} \implies o^i_j$, or equivalently $\overline{o}^i_j \implies \overline{o}^i_{j+1}$, that all literals $\overline{o}^i_j$ are implied by unit propagation from $o^i_{j'}$ with $j' < j$. Therefore, MINISAT will not return a core that contains o^i_j if $o^i_{j'}$ is in the assumptions. This means that OLL$'$ is identical to OLL given these implementation details. By inspection of the code of RC2, we can confirm that it does indeed use this order of assumptions with MINISAT, and therefore implements OLL$'$.

4.2 Cores and Hitting Sets of OLL

In the following, we overload notation that we have used already for PMRES, but we use them now in the context of OLL$'$, with the same meaning: $H^i_R, \mathcal{C}^i, \mathcal{C}^i_\cup$.

▶ **Lemma 13.** *There exists a set $\mathcal{C}^i$ such that m^i is a core of H^i if and only if for each $c \in \mathcal{C}^i$, c is a core of ϕ.*

Proof Sketch. We observe that $o^i_j = \bigvee_{S \subseteq m^i, |S| > j} (\wedge_{b \in S} b)$, therefore it is a monotone function of the inputs of the core. The entire formula constructed by OLL is therefore also monotone. We show the result using a similar variable forgetting argument as we did in lemma 2. ◀

The proofs of Lemma 3, Observation 4, and Corollary 5 transfer to OLL$'$ immediately. These establish that the WPMS instance $\langle H^i_R, cost^i \rangle$ encodes the minimum hitting set problem over $\mathcal{C}^i_\cup$, where the cores are derived as described in lemma 13 this time.

In order to show that OLL$'$ does compute minimum hitting sets at each iteration for PMS, we have to prove the equivalent of lemma 6.

▶ **Lemma 14.** *If W is a PMS instance, $H^i_R\mid_0$ is satisfiable for all iterations i of OLL$'$.*

Proof Sketch. The following invariant holds in OLL$'$: for each meta m^i, there exists $0 \le k < r^i$ such that $w(o^i_{k'}) = 0$ for all $k' \le k$ and $w(o^i_{k'}) > 0$ for all $k' > k$. Therefore, any assignment that sets $o^j_{k'}$, $k' < k$, to true can be extended by setting $o^j_{k''}$ to true as well for all $k'' < k'$ and exactly k' variables of m^i, so that all sum constraints of iteration i are satisfied.

From there, we use the same argument as we did in the proof of lemma 6 to show that, given an assignment to the variables of the metas $m^j, \dots, m^i$, $j < i$, we can extend to an assignment to the variables of m^{j-1} because any two sum constraints from different iterations sum over disjoint sets of variables. ◀

As was the case for the corresponding lemma in PMRES, Lemma 14 says nothing about instances with non-uniform weights.

4.3 OLL and Linear Programming

We prove the equivalent of theorem 9 for OLL$'$.

▶ **Theorem 15.** *There exists an integer linear program ILP^i_P which (1) is logically equivalent to the minimum hitting set problem with sets $\mathcal{C}^i_\cup$, (2) has size polynomial in $|H^i_R|$, and (3) whose linear relaxation has an optimum which matches that derived by OLL$'$.*

Proof. We construct a WCSP P^i. Its linear relaxation, the local polytope $LP(P^i)$, is the LP we want. Let $P^0 = \emptyset$. At iteration i, where the core discovered is $\{b^i_1, b^i_2, \dots, b^i_{r^i}\} \subseteq S^{i-1}$, P^i is defined as P^{i-1} and additionally the following variables and cost functions:

- 0/1 variables $b^i_1, \dots, b^i_n$, $o^i_1, \dots, o^i_{r^i-1}$, corresponding to the propositional variables of the same name in W^i.

- Unary cost functions with scope b_i for each $b_i \in vars(W^0)$, with $c(b_i, 0) = 0, c(b_i, 1) = c^0(b_i)$
- A variable O^i with domain $[0, r^j]$, with $c(O^i, 0) = \infty$ and $c(O^i, j) = 0$ for all $j \in [1, r^i]$.
- A decomposition of the sum constraint $\sum_{j\in[1,r^i]} b^i_j = O^i$, as described by Allouche et al. [1].
- Binary cost functions with scope $\{O^i, o^i_j\}$, for all $j \in [1, r^i - 1]$ where the tuples $\{j', 1\}$ and $\{j'', 0\}$, for all $1 \leq j' < j < j'' < r^i$, have infinite cost, and the rest have cost 0. These encode the constraint $o^i_j \iff O^i > j$.

As before, the equivalence of P^i and H^i_R is immediate. We show that there exists a reformulation of P^i that yields the same costs as the weights computed by OLL$'$, as well as the same lower bound. The latter relies on previous results [1], which imply that, we can move cost $w^i_{\min}$ from $b^i_1, \ldots, b^i_n$ to O^i, so that we have $c(O_i, j) = jw^i_{\min}$. Since $c(O^i, 0) = \infty$, we can apply $prj_0(O^i, w^i_{\min})$. Finally, we can apply $ext(O^i = j', \{O^i, o^i_j\}, w^i_{\min})$ for all $j' \geq j$, followed by $prj(\{O^i, o^i_j, o^i_j, w^i_{\min})$. Once we complete this for all $j \in [1, r^i]$, there is no cost in O^i, and each o^i_j has cost $w^i_{\min}$, as required. ◀

5 Connection to the Sherali-Adams hierarchy

The Sherali-Adams hierarchy of linear relaxations [28] of a 0/1 integer linear program is a well known construction for building stronger relaxations. At its k^{th} level, it uses monomials of degree k and it is known that the level n relaxation (where n is the number of variables in the ILP) represents the convex hull of the original ILP, meaning that it solves the ILP exactly. On the flip side, the size of the relaxations grows exponentially with the level of the hierarchy, meaning that even low level SA relaxations tend to be impractical.

Formally, we derive the k^{th} level SA relaxation as follows. Let $SA^u_0(LP) = LP$, the linear relaxation of the integer program. First, we define the set of multipliers $M_k = \{\prod_{i\in P_1} x_i \prod_{i\in P_2}(1 - x_i) \mid P_1, P_2 \subseteq [1, n], |P_1 \cup P_2| = k, P_1 \cap P_2 = \emptyset\}$, i.e., the set of all non-tautological monomials of degree k, using either x_i or $(1 - x_i)$ as factors. We then multiply each constraint $c \in LP_0$ by each multiplier $m \in M_k$, simplify using $x^2 = 1$ and $x(1 - x) = 0$, and finally we replace each higher order monomial by a single 0/1 variable to get $SA^u_k(LP)$.

In this description, SA^u_k does not contain the variables and constraints of LP or any SA^u_j, $j \in [1, k-1]$. Here, we use instead $SA_k(LP) = \cup^k_{i=0}(SA^u_k(LP) \cup cns(k)$, where $cns(k)$ are constraints which ensure consistency between the variables at different levels, i.e., do not allow $x_ix_j = 1$ and $x_i = 0$ at the same time.

To show the connection with PMRES, we define the *depth* measure for variables and, by extension, cores and formulas. The set the depth of all variables appearing in W^0 to be 0, and we write $dp(b_j) = 0$, for $b_j \in vars(W^0)$. Consider a meta m^i. We define $dp(f^i_j) = \max_{b_j\in m^i} dp(b_j) + 1$ for all $j \in [1, r^i - 1]$, and similarly for $d^i_j, j \in [1, r^i - 1]$. With an overload of notation, we also write $dp(m^i) = dp(f^i_1)$. Finally, at iteration i, we write $dp(W^i) = \max_{j\in[1,i]} dp(m^j)$. In words, the depth of a variable of the original instance has depth 0, the variables introduced by a meta are one level deeper than variables that appear in the meta, the depth of a meta is the same as that of the variables it introduces, and the depth of the instance at iteration i is the deepest meta PMRES has discovered.

The result of this section, is that $LP(P^i)$, the linear relaxation that achieves the bound computed by PMRES, is a subset of the $2^{dp(W^i)}$ level Sherali-Adams relaxation of a specific linear formulation of the hitting set instance $\mathcal{C}^i_\cup$.

▶ **Theorem 16.** *The variables f_j^i with $dp(f_j^i) = k$ are defined as a linear expression over variables of at most the level 2^k SA relaxation of the hitting set problem over $\mathcal{C}_\cup^i$.*

Proof. By induction. It holds for variables with depth 0, since they are variables of the original formula. Assume that it holds for variables of depth $k-1$.

The main observation is that, since $f_j^i = b_j^i \wedge d_j^i$, we can write it as $f_j^i = b_j^i d_j^i$, i.e., replace the conjunction by multiplication, which is valid for 0/1 variables. Then, since $d_j^i = b_{j+1}^i \vee \ldots \vee b_{r^i}^i$, we can write it as $d_j^i = \max(b_{j+1}^i, \ldots, b_{r^i}^i)$. The max operator is a piecewise linear function, so this expression is linear. Finally, we replace d_j^i in the definition of f_j^i to get $f_j^i = \max(b_j^i b_{j+1}^i, \ldots, b_j^i b_{r^i}^i)$. Recall that $dp(b_l^i)$ for $l \in [j+1, r^i]$ is at most 2^{k-1}, so f_j^i can be written as a linear expression over monomials of degree at most 2^k, since it multiplies two variables which are themselves a linear expression over monomials of degree at most 2^{k-1}. ◀

Theorem 16 reflects the already known connection between Max-Resolution and the Sherali-Adams hierarchy in the context of proof systems for satisfiability [17]. Moreover, it is known that the k^{th} level of the Sherali Adams hierarchy based on the *basic LP relaxation* (BLP) of a CSP, another name for the local polytope LP, establishes k-consistency [29].

Theorem 16 is fairly weak. The upper bound is extremely loose and there is no lower bound. It is useful, however, as it suggests that discovering a meta of depth k involves potentially proving 2^k-inconsistency. It also hints towards minimizing the maximum degree of monomials entailed by a meta as a metric for choosing among different potential metas.

In the greater context of PMRES compared to IHS, one way to interpret the result of this section is that the two algorithms are instantiations of the same algorithm: they are both implicit hitting set algorithms, but where IHS extracts a single core at a time and offloads the hitting set computation to a specialized solver, PMRES shifts the burden to the SAT solver to not only extract cores, but discover a higher level relaxation so that the hitting set problem can be solved in polynomial time.

6 Discussion

6.1 PM1

The results of section 3.3 have of course already been shown for PM1 [7, 25]. The result we have shown here that is not shown for PM1 is the existence of a compact LP that computes the same bound as PM1. It is not easy to see how the results of section 3.4 could transfer. For PMRES and OLL, H_R^i logically entails all the implied cores. This allows us to create an ILP representation of the hitting set problem immediately, and then strenghten the LP relaxation using higher order cost functions to achieve the same bound. But for PM1, cores are solutions of a linear system, so it is not immediately obvious even how to create an ILP representation of the hitting set problem without enumerating the (potentially exponentially many) cores of the original formula.

6.2 Practical Implications

Besides revealing a tight connection between the operation of IHS and core-guided algorithms, there are potential practical implications, in particular from theorem 9. We first observe that the linear program used to prove theorem 9 is linear in the size of H_R^i, hence the size of the LP is not too great. Moreover, it can be further reduced by noting that, in order to replicate the bound of PMRES, the dual variable corresponding to several primal constraints is always zero. Therefore, they can be removed from the LP without affecting the bound. After that, the LP can be further simplified by removing variables that appear in only 1 constraint and forgetting (in the sense of the knowledge compilation operation of forgetting) variables that appear in only two constraints. In this way, the LP is reduced to contain only the d and f variables, and uses r^i constraints to relate them. In the running example, upon discovering the core $\{b_1, b_2, b_3, b_4\}$, the LP needs only the following constraints to satisfy the requirements of theorem 9:

$$
\begin{aligned}
b_1^1 - f_1^1 - d_1^1 &= 0 \\
b_2^1 - f_2^1 - d_2^1 + d_1^1 &= 0 \\
b_3^1 - f_3^1 + b_4^1 + d_2^1 &= 1
\end{aligned}
$$

We omit the details of this mechanical reduction of the LP. But this suggests that the LP of theorem 9 is not just a theoretical construct, but a practical way to replicate the reasoning of PMRES. This allows a solver which runs PMRES until some heuristic condition is met, then passes its progress to IHS using theorem 9 to represent the hitting set problem and the lower bound. In the other direction, a solver can run IHS, then solve the hitting set problem once with PMRES to construct H_R^i, then continue solving starting from $\left\langle H_R^i \cup H, w^i \right\rangle$, in order to simplify solution of the ILP. However, running the two algorithms in sequence is the simplest form of combining them. Presumably, the greatest performance can be gained by an even deeper integration, using the LP to communicate progress.

7 Conclusion

We have narrowed the gap between implicit hitting set and core-guided algorithms for MaxSAT. We have shown that the core-guided algorithms PMRES and OLL, the latter of which is the basis for the winning solvers of some recent maxsat evaluations, implicitly compute a potentially exponentially large set of cores of the original MaxSAT formula at each iteration and a minimum hitting set of those cores under some conditions. Moreover, we showed that they build a WPMS instance which is logically equivalent to the minimum hitting set problem over those cores and can therefore be seen as a compressed, polynomial sized, encoding of that problem. In addition, we showed how this problem is solved: by generating a subset of a higher level of the Sherali-Adams linear relaxation of that hitting set problem. These results open up the possibility for tighter integration between PMRES and IHS.

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